**Quantum Mean Field Theory Fluctuations**

[might check out Condensed Matter/Metals/Interacting Electrons/Exchange/Thermal Properties file for more on these techniques applied to spins]

Now we’re going to take our model and make systematic corrections to the mean field result. So recall we found,



where,



BTW, since we’re now considering fluctuations, our Tc won’t be Jz anymore. So I’ll avoid treating the two synonymously. Maybe I’ll call the latter Tcm, i.e. Tcm = Jz. Okay let’s put this in dimensionless form. So we’ll change variables x → xa. And in the second line, rename the fields φ(ax) → φ(x) and j(ax) → j(x),



and we recalled that n0 = 1/ad. And as T → Tcm = Jz, Kz → 1, so we could say,



where,



to first order in T-Jz. Also remember from the last file that we have, to first non-zero order in T – Jz:



So in this limit we can identify the magnetization with the average φ.

**Analyzing Z[T,h(x)] model near criticial point and fluctuations**

It seems a word is in order about the distinction between what we’ve done and what we’re about to do. So we’ve done an inhomogeneous mean field analysis of the action above in a previous file. It consisted of a saddle point approximation of the above action, effectively treating the saddle point as the only significant contribution to Z. And for practical purposes we approximated the non-linear saddle point equation by making the replacement um(x)3 → um2m(x), where m is the homogeneous magnetization. And once we did this, we found all of our MF exponents, α = 0, β = ½, γ = 1, δ = 3, ν = ½, η = 0.

What we’re going to do below is a little different. We’re not going to work out the saddle point equation per se´; instead we’re going to do a perturbative analysis of the action, in powers of u, expanding around the point φ = 0. So we’ll be incorporating fluctuations in way that we did not with MFT. We’ll be adding up contributions of more ‘paths’ to Z than we do with the just the saddle point approximation. But this will only work for small φ, and one might say that the MF approach is still a better way to incorporate a non-zero-ish uφ4 term than simply stopping at some finite order of perturbation theory. So we won’t stop there. We’ll go on to do a self-consistent Hartree-Fock approximation on the self-energy, which is equivalent to summing to all orders of perturbation theory a certain class of diagrams. This will enable us to get beyond the perturbative requirement that φ(x) ≈ 0. And we’ll find that it kind of reproduces our prior non-linear saddle point equation from (see Ising Weiss MFT – inhomogeneous) that file. But in the variable φ(x), rather than m(x) – the important distinction being that φ(x) is fluctuatey, but m(x) = <φ(x)> is not. And along with that, it will come with the stipulation that we basically make the approximation uφ(x)4 = uφ(x)2<φ(x)2>, whereas the saddle point analysis basically says uφ(x)4 = um(x)4 = u<φ(x)>4. So the self-consistent Hartree-Fock expansion will do a better job of capturing the effect of fluctuations, which will make all the difference between the Hartree-Fock results for the critical exponents, etc., and the MFT results for the exponents. Anyway, so we start with:



and the FD rules (see path integral file, and note 6/4! = 1/4):

Diagram

Description automatically generated

We’ll actually put these in Fourier space. And once we do, then we sum (1/Ld)Σk or integrate ∫ddk/(2π)d over all momenta. Well might note that since we factored the lattice spacing, *a,* out of our x coordinate in the action, the sum would really be (1/N)Σk since k is now unitless. Maybe see Path Integrals folder, Fourier transform folder, or Quantum field theory folder, etc. And just to be the clear, the Green’s function would satisfy:



And it follows that the Fourier transform of G0 would be given by:



Now we can get the magnetization, for instance. This would be:



So,



The generalized susceptibility/Green’s function would be given by:



where φ0 = <φ(x)> = m(x) and is presumed to not depend on x. In the disordered phase φ0 is zero, in which case this would just be β<φ(x)φ(x´)>. In any event. This guy is just β × the exact two-point GF.



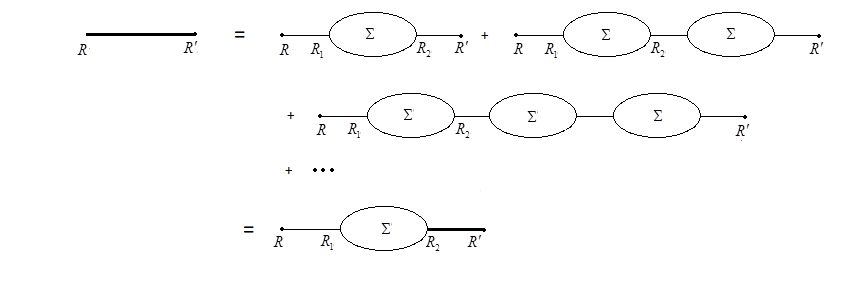
We can interpret this as the average fluctuation response of a spin at x´ to a fluctuation of a spin at x. We can get a lot of critical exponents from G. Recall its general form,



And also recall (Thermodynamics folder/Critical Exponents) that from the Fourier transform of χ(r-r´), we have the uniform susceptibility, and another critical exponent,



as well as the critical temperature I guess. And of course with these critical exponents in hand, we can obtain all others using the Widom scaling hypothesis stuff (see Thermodynamics/Critical Exponents), if we include the hyperscaling relation as well. So the diagrammatic expansion of G would just be the sum of all fully connected diagrams. Maybe see the Collective Excitations file in the electron-electron interaction folder (Condensed Matter) for more on that point. Since it’s connected we can introduce a self-energy, and write, diagrammatically:



and in symbols,



So,



As stated, the k = 0 term would be equivalent to the uniform susceptibility χ(T)/β. And we could determine both the critical point Tc and the exponent γ by approximating r – Σ(r,0) near its root.



where rc is the pole. Explicitly this is (letting kB = 1):



So we have:



where Jz is of course the mean field critical temperature Tcm. Well this also follows plainly from the definition of r. Whatever…

**Zeroth order calculation of χ(q)**

The zeroth order calculation of χ(q) basically corresponds to the MFT approach, in region where φ = 0. So we’re approaching the transition from the disordered side. And to this order, we have:



And we have:



So we just get the usual Tc = Tcm = Jz and γ = 1 stuff from mean field theory. Now we’ll go on to see what contribution fluctuations about the mean field result make to these values.

**First order calculation of χ(q)**

The first order term in Σ is:



and it is (remember ½ because of coincident propagator, and it doesn’t depend on k as can see from the closed loop)



where Ωd = (1, π, 4π, …) is the surface area of a d-dimensional unit sphere, Sd = Ωd/(2π)d, Λ is the ultraviolet cutoff for k ~ 2π/a where a is the lattice spacing (but at the top of the page, we switched to units where x → xa, which means a ~ 1, so the ultraviolet cutoff is like k ~ 2π. When we set k = 0 (but there isn’t one) we see the renormalized r.



So we have:



We would like to approximate this near the root:



So first let’s solve for the root.



Well, I don’t want to find the root. But important thing is that whatever the root is, (r) goes from negative to positive (or vice versa) before and after rc. So it can be expanded linearly about the root. And so γ = 1 basically. Note this result is independent of u. So since the first order result didn’t change the exponent, it isn’t clear whether a second order result will. Still, we can improve things by going to infinite order in perturbation theory within a certain subset of diagrams. This is the self-consistent HF approximation.

**Self-consistent HF first order approximation to χ(q)**

So let’s use a self-consistent approximation. Going back to the action,



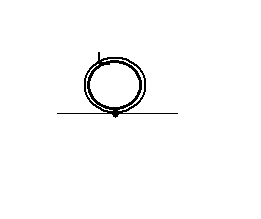
we can self-consistently calculate G by rewriting φ4 as:



This is equivalent to what we’ll do below, instead – a self-consistent ‘Hartree-Fock’ calculation of the self energy. So going back to:



We’ll use the full GF in the Σ(r) term. So,



and the thick line is the exact propagator, GF. In that case we have the self-consistent equations,



Before proceeding, let’s point out that we had a similar equation in the Ising Weiss MF – inhomogeous file. We found:



If translate χ → βG, and use the fact that near the (mean field) critical point βJz → 1, as well incorporate the change of independent variables which x → ax, we’d have:



By comparison, in position space, our self consistent equation would read, rather,



where <δφ(x)δφ(x)> = G(x = x´). So basically, we are replacing the magnetization average squared with the average squared magnetization, which should be more faithfull to the fluctuations present in the system. So let’s proceed and solve for our self-energy.



Observe how Σ will just be a parameter in the integral. So we can solve for it exactly, in principle. Recalling our definition, = r – Σ(r), we get,



Again,



and so our equation comes to:



which is:



Now still, we want to write ~ (r – rc)γ. But now we have a self-consistent equation to solve for it. First we determine the critical temperature. This is when = 0, and as can see from equation, this occurs when r = rc = -cΛd-2. What is this? Recall formula for Tc above.



So we have:



So the correction to the critical temperature is just given by that. That was easier. First we’ll note that the critical temperature is lower than its mean field value. This would be because the fluctuations we’re capturing militate against order, and so we need to go to lower temperatures to restore it. Also note that the critical temperature goes back up with dimension. This would be because fluctuations are less effective in higher dimensions, as the mean is an increasingly good representation of the actual value as the ‘sample’ size increases. Accordingly, in the very large d limit, it’s given by just the mean field value. Conversely, we can see that Tc → 0 as d → 2. This seems to indicate that there is a lower critical dimension (d = 2 in this case), below which the transition cannot occur, because fluctuations are so violent that order cannot be established. This was not evident in our prior MFT analysis. But now we want to determine the critical exponents. So we want to expand in powers about this value.

**dimension d > 4**

To do this we note that if d > 4 then near the critical point (small ), the first term in the equation dominates over the d/2-1 and we have,



and so γ is still 1, and we only have a correction to Tc. So the fluctuation corrections don’t change the critical exponents from their MFT values. This is what we should expect, as by the law of averages MFT works for d > dupper critical.

**dimension 2 < d < 4**

But for 2 < d < 4, it is the d/2-1 term which dominates for small , and in addition to a TC correction, there is a critical exponent ‘correction’, as now we have:



and then the critical exponent γ changes to γ = 1/(d/2-1), which in 3D would be 2 (its actually around 1.33 I believe). What is the region where the first term dominates?



So the MF result/behavior dominates for,



and it’s only very close to the critical point that the fluctuations take over. But still, near the critical point it appears that these fluctuations are too strong for MFT to be an accurate approximation. But apparently, even taking account of fluctuations at the SCHF level doesn’t suffice to procure agreement with experimental values of critical exponents.

**dimension d ≤ 2**

We’ll also observe that things blow up at d = 2, and thereafter for lower d’s the critical exponent becomes negative, which doesn’t make physical sense. So when we include fluctuations, this casts doubts on the actuality of a phase transition for dimensions less than 2. And in fact we ‘know’ there isn’t one for dimensions less than two. And we did explicitly see this for the XY model and d = 1.

**General results**

So the general result is that for d < 2, we probably don’t have a phase transition at all (can’t be *sure* at this stage), as fluctuations dominate near the critical point, giving rise to negative critical exponents that physically shouldn’t be negative, etc.

For 2 < d < 4, the fluctuations dominate for |T – Tc| → 0 (on a scale determined by ) and we have a correction to TC and a correction to γ, and so presumably all the other exponents too. But as we saw at least for γ, the correction we find from SCHF isn’t actually better than the MF result. For |T – Tc| >> 0 (on a scale determined by ) the mean field term dominates and everything goes back to mean field theory with a correction to Tc, but not to γ. So in other words, χ(T) would look like the old MF χ(T) except for being shifted to the new TC­ and having different power law behavior only very close to the new TC. Thus MFT can do very well even for systems close to the critical point, just not in the very close vicinity, and it will get the exponents quite wrong as well, in that very close region.

For d ≥ 4, mean field theory dominates everywhere, and is basically correct.

So we can conclude two things. The failure of the perturbative approach, where we stop at the first order term in the expansion implies that, at least close to the critical point, the uφ4 term *isn’t a small perturbation* on the rest of the model when we’re looking at points near the critical point, where long wavelength fluctuations dominate. While <φ(x)>4 is small, we cannot conclude that random variable φ(x)4 is small, i.e, that <φ(x)4>, or <φ(x)2>, etc., are small. And the failure of the self-consistent HF approach implies that even going to all orders of perturbation theory with a single subset of diagrams may not suffice. So for instance we cannot approximate <φ4> = <φ2>2. Many different classes of diagrams, besides just HF, might have to go into a self-consistent self-energy approach to correctly reproduce the fluctuations. What these might be is unknown at moment. The HF approach replaced φ(x)2 with <φ(x)2>, which is better than the mean field approach which replaced φ(x) with <φ(x)>. But still I guess that’s not good enough.

Perhaps the uφ4 term is a small perturbation for small wavelength fluctuations because you can treat it perturbatively for some cases and get good results (I think). So what is needed is a way to isolate the long wavelength fluctuations and see how they interact with each other. Then perhaps we can study that model. This is what RG allows us to do.